STRAIN AND THE TRANSFORMATION OF STRAIN

INTRODUCTION - DEFORMABLE BODY MOTION

1) Rigid Body Motion

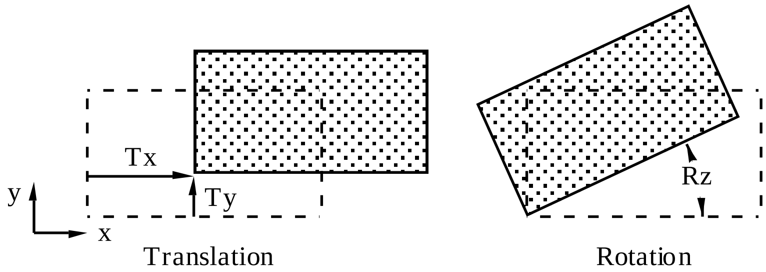

Figure 1: General Plane Motion (Translation and Rotation)

Figure 1 shows the general plane motion of a rigid body consisting of a translation, where all points on the body have the same displacement (T_x,T_y), and a rotation, where all lines on the body have the same rotation (R_z). In rigid body motion, no line changes length nor is there a change in the angle between any two lines.

2) Deformation and Strain

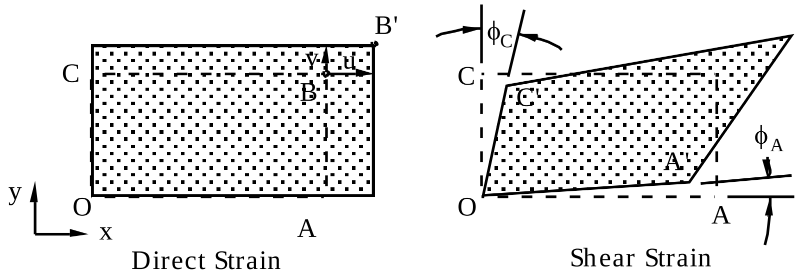

Figure 2: Deformable Body Motion (Strain)

Figure 2 shows the plane deformation of a body. This motion involves a change of volume, direct (normal) strain, and a change of shape, shear strain. Direct strain represents a change in length of a line and shear strain represents a change in angle between lines. In Figure 2, direct strain is demonstrated by the change in length of line OB to OB' and shear strain is demonstrated by the change in angle COA to C'OA'.

As strain displacements may vary from point to point (and line to line) strain is defined as a measure of relative displacement. e.g. In Figure 2 the average direct strain is given by u/OA in the x direction and v/OC in the y direction. The average shear strain is given by $(\phi_A+\phi_C)/\angle COA$.

3) Energy

When a deformable body is acted upon by forces it moves. This motion may consist of both rigid body motion and deformation. Rigid body motion is associated with kinetic energy and gravitational potential energy and deformation is associated with strain potential energy.

INFINITESIMAL STRAIN

1) Direct (Normal) Strain

Figure 3 shows a line element in a body and the change in length of this line represents direct strain. Engineering direct strain is defined as:

$$\frac{\text{change in length}}{\text{original length}} = \frac{\Delta L}{L_o}$$

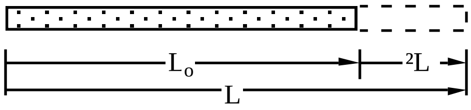

Figure 3: Direct Strain

In general, strain is not uniform over a finite length so engineering direct strain is defined by the extension of an elemental length. i.e.

$$\varepsilon = \left(\frac{\Delta L}{L}\right)_{L \to 0}$$

If u and v are the displacements in the x and y directions, the engineering direct strain is defined as:

$$\varepsilon_x = \frac{du}{dx} \qquad \varepsilon_y = \frac{dv}{dy} \quad \text{..}(1)$$

2) Shear Strain

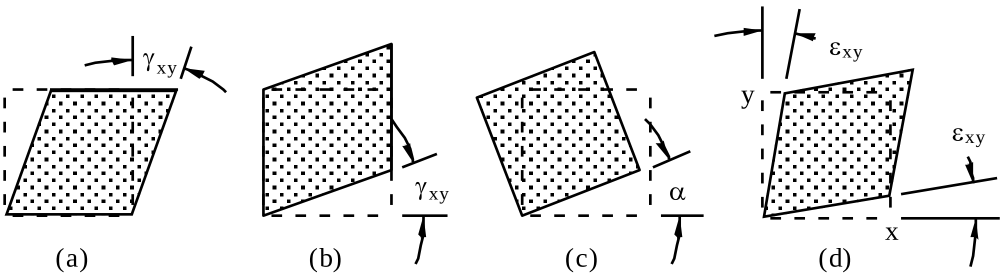

Figure 4: Shear Strain and Rotation

Figure 4 shows deformations involving the rotation of lines in an element. Figure 4a and 4b show two examples of shear strain but in Figure 4c, although there is rotation of a line, there is no shear action. There is thus a need to distinguish between the shear rotation of a line and the rigid body rotation of a line. This is achieved by defining shear strain by the change in angle between two orthogonal lines as shown in Figure 4d.

The angle ε_{xy} is known as *tensor shear strain* and γ_{xy} is known as *engineering shear strain* where:

$$\gamma_{xy} = 2\varepsilon_{xy} \text{..}(2)$$

Note that Figure 4a represents tensor shear strain $\varepsilon_{xy} = \gamma_{xy}/2$ with a clockwise rotation of $\alpha = \gamma_{xy}/2$. Figure 4b also represents tensor shear strain $\varepsilon_{xy} = \gamma_{xy}/2$ but with a counter-clockwise rotation of $\alpha = \gamma_{xy}/2$. Shear strain is positive when there is an extension of the diagonal with positive slope. The shear strain shown in Figure 4d is positive.

PRINCIPAL STRAINS

When a body deforms there will always be elements defined by orthogonal axes which do not undergo shear deformation. The axes defining these elements are the *principal axes* and the strains in these directions are the *principal strains*. It can be shown that the principal strains are the maximum and minimum direct strains.

e.g. In Figure 5 the square aligned with the xy axes (ABCD) is subjected to pure shear but the diagonals of this square do not rotate. Therefore the sides of a square aligned with the 12 axes (abcd) do not rotate and the abcd square is subjected to normal strains but not shear strain. As shown below, it is possible to use these principal strains to obtain the strains in any other direction.

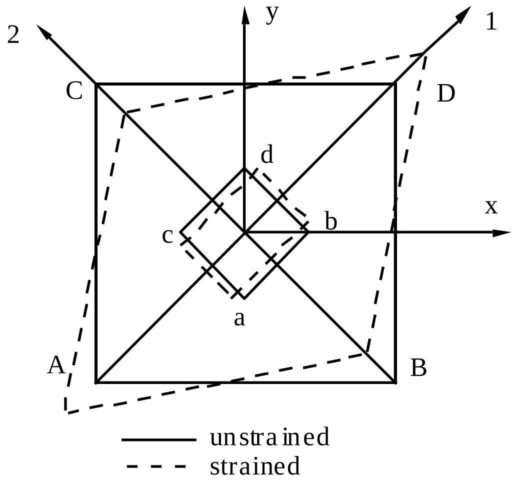

Figure 5: Pure Shear and Principal Axes

Transformation of Strain

In Figure 6, line OB on the element aligned with the (principal) 12 axes extends and rotates to OB'. For this line, the shear and normal strain in the x directions are thus given by:

$$\varepsilon_{xy} = \frac{BC}{OB} \qquad \varepsilon_x = \frac{CB'}{OB}$$

In the 12 directions, the (principal) strains are given by:

$$\varepsilon_1 = \frac{BA}{OD} \qquad \varepsilon_2 = \frac{AB'}{DB}$$

The displacement of OB can therefore be written in terms of the strains in both the xy and the 12 directions:

i.e. vector $\overline{BB'} = \overline{\Delta_1} + \overline{\Delta_2} = \overline{\Delta_x} + \overline{\Delta_y}$

where:

$$\Delta_1 = BA = OD\,\varepsilon_1 = OB\cos\theta\,\varepsilon_1$$
$$\Delta_2 = AB' = DB\,\varepsilon_2 = OB\sin\theta\,\varepsilon_2$$

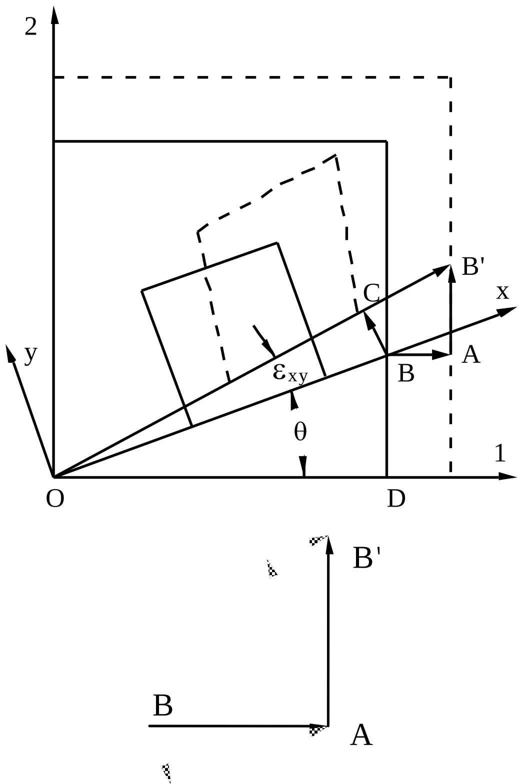

Figure 6: Transformation of Strain

and:

$$\Delta_x = CB' = OB\,\varepsilon_x = (\Delta_1)_x + (\Delta_2)_x = OB\cos^2\theta\,\varepsilon_1 + OB\sin^2\theta\,\varepsilon_2$$
$$\Delta_y = BC = OB\varepsilon_{xy} = (\Delta_1)_y + (\Delta_2)_y = -OB\cos\theta\sin\theta\varepsilon_1 + OB\sin\theta\cos\theta\,\varepsilon_2$$

Leading to:

$$\varepsilon_x = \varepsilon_1\cos^2\theta + \varepsilon_2\sin^2\theta \quad \text{...}(3)$$
$$\varepsilon_{xy} = -(\varepsilon_1 - \varepsilon_2)\cos\theta\sin\theta \quad \text{...}(4)$$

Substituting

$$\cos^2\theta,\ \sin^2\theta = \frac{1}{2}(1 \pm \cos 2\theta), \qquad \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$$

gives:

$$\varepsilon_x = \frac{\varepsilon_1+\varepsilon_2}{2} + \frac{\varepsilon_1-\varepsilon_2}{2}\cos 2\theta \text{ ......(5)}$$

$$\varepsilon_{xy} = -\frac{1}{2}(\varepsilon_1-\varepsilon_2)\sin 2\theta \text{ ......(6)}$$

As with the transformation of stress, Eqns. (5) and (6) are the parametric equations of a circle known as Mohr's circle of strain.

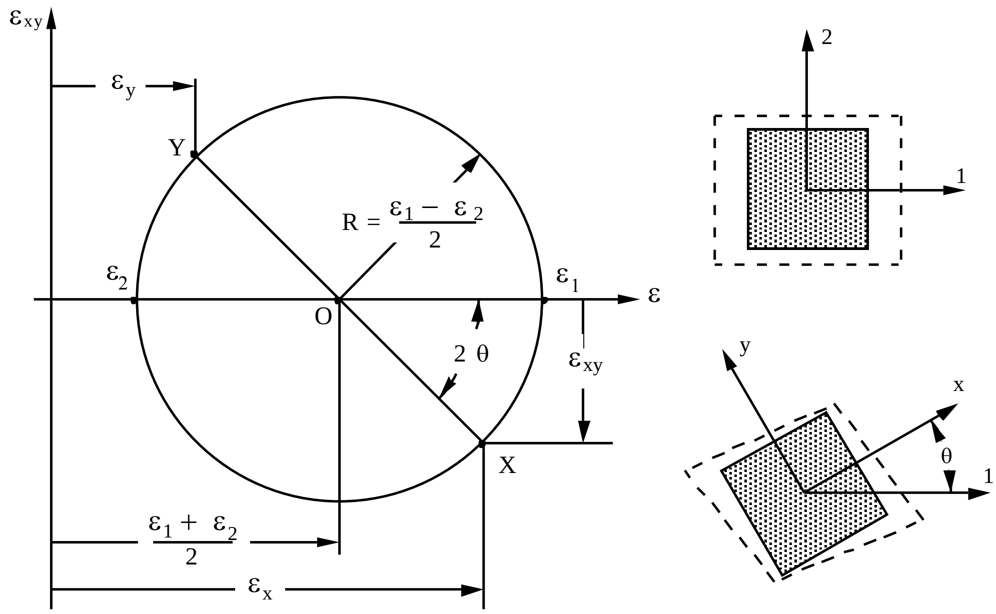

Figure 7: Mohr's Circle of Strain

From Figure 7 it can be seen that the maximum shear strain occurs on planes which are ±45° from the principal planes and that the maximum shear strain is given by:

$$\varepsilon_{xy_{max}} = \frac{(\varepsilon_1-\varepsilon_2)}{2} \text{ ......(7)}$$

On the planes of maximum shear both direct strains are $(\varepsilon_x + \varepsilon_y)/2$. Mohr's circle also shows that for all orthogonal planes, the sum of the direct strains is a constant:

$$\varepsilon_x + \varepsilon_y = \varepsilon_1 + \varepsilon_2 \text{ ......(8)}$$

HOOKE'S LAW AND ELASTIC CONSTANTS

1) Young's Modulus and Modulus of Rigidity

Consider a body subjected to simple tension where:

$$\sigma_1 = \sigma_x \qquad \sigma_2 = \sigma_3 = 0$$

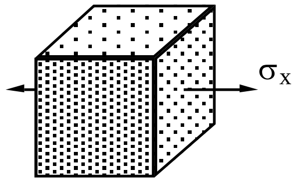

Figure 8: Simple Tension

From Hooke's law for plane strain (and a linear material) the principal strains are:

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2 - \nu\sigma_3\right) = \frac{\sigma_x}{E}$$

$$\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - \nu\sigma_3 - \nu\sigma_1\right) = \frac{-\nu\sigma_x}{E}$$

and the maximum shear strain is:

$$\varepsilon_{xy\,max} = \frac{\left(\varepsilon_1 - \varepsilon_2\right)}{2} = \frac{\left(\sigma_x - \left(-\nu\sigma_x\right)\right)}{2E} = \frac{1+\nu}{2E}\sigma_x = \frac{\gamma_{xy\,max}}{2} = \frac{\tau_{xy\,max}}{2G}$$

From Mohr's circle for stress, the maximum shear stress is given by:

$$\tau_{xy_{max}} = \left(\sigma_1 - \sigma_2\right)/\ 2 = \sigma_x /\ 2$$

so we have: $$\frac{1+\nu}{2E}\sigma_x = \frac{\sigma_x / 2}{2G}$$

and hence: $$G = \frac{E}{2(1+\nu)} \quad \text{..(9)}$$

2) Dilation and Bulk Modulus

Dilation (e) is the change of volume per unit volume and bulk modulus (K) is a volume stiffness defined by uniform pressure (p)/e.

i.e. $$e = \frac{\Delta_{vol}}{Vol} \qquad K = \frac{p}{e}$$

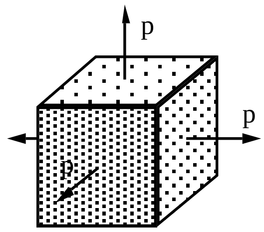

Figure 9: Uniform Pressure

When a body is subjected to a general state of stress it can be shown that the dilation is given by:

$$e = \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3\right)$$

and in a state of uniform pressure, the principal stresses are:

$$\sigma_1 = \sigma_2 = \sigma_3 = p$$

Using Hooke's law, the dilation for this state of stress is:

$$e = \frac{1}{E}\left[\left(\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right) + \left(\sigma_2 - \nu\left(\sigma_3 + \sigma_1\right)\right) + \left(\sigma_3 - \nu\left(\sigma_2 + \sigma_1\right)\right)\right] = \frac{3p}{E}\left(1 - 2\nu\right)$$

Hence:

$$K = \frac{p}{e} = \frac{E}{3(1-2\nu)} \quad \text{..(10)}$$

ELECTRICAL RESISTANCE STRAIN GAUGES

The electrical resistance, R, of a length of wire is given by $R = \rho L/A$ where ρ = resistivity, L = length and A = cross sectional area. If the length of the wire changes there will be a change in resistance and this property is used to measure strain. The most common strain gauges are manufactured from foil bonded to a non-conductive backing. The gauge is then bonded to the test material. As shown in Figure 11, gauges are typically formed by a number of "loops" which are elongated (or shortened) by the direct strain ε.

Note that the area of the foil at the ends of each "loop" is greater than the nominal area. This reduces the gauge's transverse resistance thus reducing the gauge's sensitivity to transverse strain.

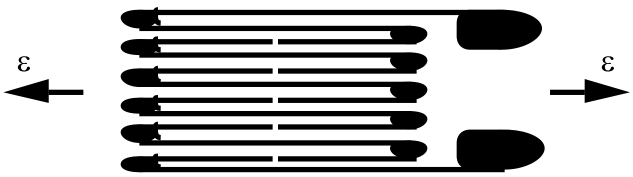

Figure 11: Resistance Strain Gauge

In addition to strain, environmental effects, especially variations in temperature, will also change a gauge's resistance. Some gauges have thermal properties that match those of the test material and these gauges do not require any allowance to be made for temperature changes. However, there is usually a limit on the range of operating temperatures and test materials for such gauges. These limitations can be overcome by the use of a "dummy" gauge. The dummy gauge is bonded to a sample of the same material as the test material and then subjected to the same environment, but not the same loads, as the "active" gauge.

The change in resistance of a gauge is therefore given by:

$$\frac{dR}{R} = \left(\frac{dR}{R}\right)_{\varepsilon} + \left(\frac{dR}{R}\right)_{T} = K\varepsilon + \left(\frac{dR}{R}\right)_{T}$$

where:

$$\left(\frac{dR}{R}\right)_{\varepsilon} = \text{change in R due to strain developed by load}$$

$$\left(\frac{dR}{R}\right)_{T} = \text{change in R due to environmental effects}$$

$$K = \text{"gauge factor"}$$

$$\varepsilon = \text{(direct) strain developed by the load}$$

Typically, the small change in a gauge's resistance is detected by a strain gauge amplifier which measures the potential difference across a "balanced" Wheatstone Bridge.

The Wheatstone Bridge

The voltage detected by the bridge is given by:

$$E = V\frac{R_1R_3 - R_2R_4}{(R_1 + R_3)(R_3 + R_4)}$$

When $R_1R_3 = R_2R_4$, E = 0 and the bridge is said to be balanced. In the balanced condition:

$$\frac{R_2}{R_1} = \frac{R_3}{R_4} = r \text{ (say)}$$

and a small change in resistance, dR, will produce a small change in voltage, dE, where:

$$\frac{dE}{V} = \frac{r}{(1+r)^2}\left[\frac{dR_1}{R_1} - \frac{dR_2}{R_2} + \frac{dR_3}{R_3} - \frac{dR_4}{R_4}\right]$$

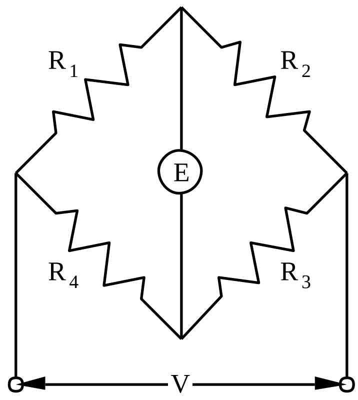

Figure 12: A Wheatstone Bridge

The Quarter Bridge

For strain gauge applications it is common to make R_1 the active gauge, R_2 the dummy gauge and R_3 and R_4 standard resistors with $R_3 = R_4$ so that r = 1. This arrangement is known as a quarter bridge whereby:

$$\frac{dR_1}{R_1} = K\varepsilon + \left(\frac{dR}{R}\right)_T; \quad \frac{dR_2}{R_2} = \left(\frac{dR}{R}\right)_T; \quad dR_3 = dR_4 = 0$$

$$\frac{dE}{V} = \frac{1}{(1+1)^2}\left[K\varepsilon + \left(\frac{dR}{R}\right)_T - \left(\frac{dR}{R}\right)_T + 0 - 0\right] = \frac{K\varepsilon}{4}$$

giving: $$(\varepsilon)_{meas} = \frac{4}{K}\frac{dE}{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}$$

The strain gauge amplifier indicates strain by measuring dE with the factor of 4 included in the output. The amplifier also requires an input value for the gauge factor. For most foil gauges the gauge factor is approximately equal to 2. If the amplifier gauge factor setting is different to the actual gauge factor, the true strain is given by:

$$(\varepsilon)_{true} = (\varepsilon)_{meas}\frac{K_{setting}}{K_{true}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(11)}$$

Half and Full Bridges

In some applications, where $\varepsilon_1 = -\varepsilon_2$, it is possible to use two active gauges (half bridge). In this configuration:

$$\frac{dR_1}{R_1} = K\varepsilon + \left(\frac{dR}{R}\right)_T; \quad \frac{dR_2}{R_2} = -K\varepsilon + \left(\frac{dR}{R}\right)_T; \quad dR_3 = dR_4 = 0$$

$$\frac{dE}{V} = \frac{1}{(1+1)^2}\left[K\varepsilon + \left(\frac{dR}{R}\right)_T - \left(-K\varepsilon + \left(\frac{dR}{R}\right)_T\right) + 0 - 0\right] = \frac{K\varepsilon}{2}$$

and the true strain is given by: $$(\varepsilon)_{true} = \frac{(\varepsilon)_{meas}}{2}\frac{K_{setting}}{K_{true}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(12)}$$

If $\varepsilon_1 = -\varepsilon_2$, it is also possible to use four active gauges (full bridge) with dR_1 and dR_3 indicating ε_1 and dR_2 and dR_4 indicating ε_2 so that:

$$\frac{dR_1}{R_1} = \frac{dR_3}{R_3} = K\varepsilon + \left(\frac{dR}{R}\right)_T ; \quad \frac{dR_2}{R_2} = \frac{dR_4}{R_4} = -K\varepsilon + \left(\frac{dR}{R}\right)_T$$

$$\frac{dE}{V} = \frac{1}{4}\left[2\left(K\varepsilon + \left(\frac{dR}{R}\right)_T\right) + 2\left(-K\varepsilon + \left(\frac{dR}{R}\right)_T\right)\right] = K\varepsilon$$

and:

$$(\varepsilon)_{true} = \frac{(\varepsilon)_{meas}}{4}\frac{K_{setting}}{K_{true}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Strain Gauge Rosettes

The complete state of strain at a point can be measured by an arrangement of three strain gauges. The most common arrangement is the rectangular rosette shown in Figure 13.

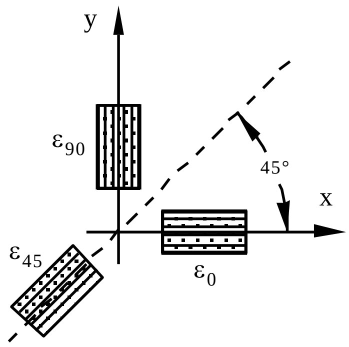

Figure 13: The Rectangular Rosette

This rosette measures three direct strains ε_0, ε_{45} and ε_{90}. If the ε_0 gauge is taken to be aligned with the x axis then the transformation of strain equation[1] gives:

$$\varepsilon_0 = \varepsilon_x \qquad \varepsilon_{45} = \varepsilon_x\frac{1}{2} + \varepsilon_y\frac{1}{2} + \gamma_{xy}\frac{1}{2} \qquad \varepsilon_{90} = \varepsilon_y$$

giving $\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_0 - \varepsilon_{90}$

Assuming plane stress conditions ($\sigma_3 = 0$) and applying Hooke's law, the strains are:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad \tau_{xy} = \frac{1}{G}\gamma_{xy}$$

Rearranging in favour of stress gives:

$$\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) \quad \sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x) \quad \tau_{xy} = G\gamma_{xy}$$

Substituting the measured values yields:

$$\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_0 + \nu\varepsilon_{90}) \quad \sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_{90} + \nu\varepsilon_0) \quad \tau_{xy} = G(2\varepsilon_{45} - \varepsilon_0 - \varepsilon_{90}) \ldots.. (14)$$

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1 $\varepsilon_{x'} = \varepsilon_x\cos^2\theta + \varepsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$